SUBCLASSES OF THE CONFORMAL ALMOST CONTACT METRIC MANIFOLDS

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A classification scheme of the conformal almost contact metric manifolds with respect to the covariant derivative of the Lee form is given. The subclasses of one basic class and their exact characterizations by the maximal subgroups of the contact conformal group preserving itself are found.

1. PRELIMINARIES. Let V be a $(2n+1)$-dimensional real vector space with almost contact metric structure $(\varphi; \xi; \eta; g)$, where φ is a tensor of type $(1,1)$, ξ is a vector, η is a covector and g is a positive definite metric such that

$$\varphi^2 = -id + \eta \otimes \xi, \quad \varphi(\xi) = 0, \quad g(\xi, \xi) = 1, \quad g \circ \varphi = g - \eta \otimes \eta.$$

The operators h and v on V are defined by
$$h = -\varphi^2, \quad v = \eta \otimes \xi.$$
The decomposition $V = hV \oplus vV$ is orthogonal and invariant under the action of $U(n) \times 1$.

2. THE SPACE OF THE TENSORS OF TYPE $(0,2)$ ON AN ALMOST CONTACT METRIC VECTOR SPACE. Let $L = V^* \otimes V^*$ denote the space of the tensors of type $(0,2)$ on $V^{2n+1}(\varphi, \xi, \eta, g)$. The metric g induces a natural inner product in L. The standard representation of $U(n) \times 1$ in V induces an associated representation of $U(n) \times 1$ in L. The operators S, A, h, v and w on L are defined by

$$SL(x,y) = [L(x,y) + L(y,x)]/2,$$
$$AL(x,y) = [L(x,y) - L(y,x)]/2,$$
$$hL(x,y) = L(hx, hy),$$
$$vL(x,y) = \eta(x).L(\xi,y) + \eta(y).L(x,\xi) - 2\eta(x)\eta(y).L(\xi,\xi),$$
$$wL(x,y) = \eta(x)\eta(y).L(\xi,\xi),$$

for arbitrary $L \in L$, x and y in V.

The traces α and β of a trensor L in L are defined by

$$\alpha = g(e^i, e^j).hL(e_i, e_j), \quad \beta = g(e^i, e^j).hL(e_i, \varphi e_j),$$

where $\{e_i\}, i = 1, \ldots, 2n+1$ is a basis of V and $\{e^i\}$ is the dual basis.

The tensors $L_i(L)$, $i = 1, \ldots, 9$ associated with an arbitrary $L \in L$ are defined

$$
\begin{aligned}
&L_1(L)(x, y) = \alpha.hg(x, y),\\
&L_2(L)(x, y) = \tfrac{1}{2}[ShL(x, y) + ShL \circ \varphi(x, y)] - L_1(L)(x, y),\\
&L_3(L)(x, y) = \tfrac{1}{2}[ShL(x, y) - ShL \circ \varphi(x, y)],\\
&L_4(L)(x, y) = \beta.hg(x, \varphi y),\\
&L_5(L)(x, y) = \tfrac{1}{2}[AhL(x, y) + AhL \circ \varphi(x, y)] - L_4(L)(x, y),\\
&L_6(L)(x, y) = \tfrac{1}{2}[AhL(x, y) - AhL \circ \varphi(x, y)],\\
&L_7(L)(x, y) = SvL(x, y),\\
&L_8(L)(x, y) = AvL(x, y),\\
&L_9(L)(x, y) = wL(x, y),
\end{aligned}
$$

(1)

for all x, y in V.

The subspaces L_i of L are determined by the conditions

$$L_i = \{L \in L : L = L_i(L)\}, i = 1, \ldots, 9.$$

Since for any $L \in L$ are valid the following equations

$$L = ShL + AhL + SvL + AvL + wL,$$

$$ShL = L_1(L) + L_2(L) + L_3(L),$$

$$AhL = L_4(L) + L_5(L) + L_6(L),$$

it is not difficult to prove the following

THEOREM 1. The decomposition

$$L = \overset{9}{\underset{i=1}{\oplus}} L_i$$

is orthogonal and invariant under the action of $U(n) \times 1$. The corresponding components of a tensor L in L are $L_i(L), i = 1, \ldots, 9$.

3. APPLICATIONS TO ALMOST CONTACT METRIC MANIFOLDS. Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact metric manifold where φ is a tensor field of type (1,1), ξ - a vector field, η - 1-form and g a Riemannian metric such that

$$\varphi^2 = -id + \eta \otimes \xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad g(\xi, \xi) = 1, \quad g \circ \varphi = g - \eta \otimes \eta.$$

The tangential space T_pM at M in $p \in M$ is an almost contact metric vector space.

Let ∇ be the Levi-Civita connection on M. The fundamental 2-form Φ is given by $\Phi(x, y) = g(x, \varphi y)$ for all tangent vectors $x, y \in T_pM$, $p \in M$. We denote

$$F = -\nabla \Phi.$$

The following 1-forms are associated with F:

$$f(z) = \sum_{i=1}^{2n+1} F(e_i, e_i, z), \quad f^*(z) = \sum_{i=1}^{2n+1} F(e_i, \varphi e_i, z), \quad \omega(z) = F(\xi, \xi, z),$$

where $\{e_i\}$, $i = 1, \ldots, 2n + 1$ is an arbitrary orthonormal basis of T_pM, $z \in T_pM$ and $p \in M$. We denote $\Omega = \mathfrak{X}^*M$ – the space of 1-forms on M and L_2^0M – the space of tensor

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fields of type (0,2) on M. The covariant derivative of a form defines the map

$$L : \theta \in \Omega \longrightarrow L(\theta) = \nabla\theta \in L_2^0 M.$$

Thus the subspaces L_i in the decomposition of the space L given in Theorem 1 induce the corresponding subspaces Ω_i of the space Ω. More precisely

(2) $$\Omega_i = \{\theta \in \Omega : L(\theta) = L_i(\theta)\},$$

where $L_i(\theta)$ are the components of $L(\theta)$ in L_i , i.e. $L_i(\theta)_p = L_i(L(\theta)_p)$. The subspaces corresponding to $L_i \oplus L_j$ will be denoted by $\Omega_i \oplus \Omega_j$, $i, j = 1, \ldots, 9$.

The following proposition is well known.

PROPOSITION 1. Let $\theta \in \Omega$. Then

i) $L(\theta)$ is a symmetric tensor field iff θ is closed;

ii) $L(\theta)$ is an anisymmetric tensor field iff the dual vector field corresponding to θ is a Killing vector field.

From Theorem 1. and Proposition 1. it follows

PROPOSITION 2. Let $\theta \in \Omega$. Then

i) θ is closed iff $\theta \in \Omega_1 \oplus \Omega_2 \oplus \Omega_3 \oplus \Omega_7 \oplus \Omega_9$;

i) θ^* is Killing vector field iff $\theta \in \Omega_4 \oplus \Omega_5 \oplus \Omega_6 \oplus \Omega_8$,

where θ^* is the dual vector corresponding to θ.

4. CLASSIFICATION SCHEME FOR THE CLASS W_1 OF ALMOST CONTACT METRIC MANIFOLDS.   Using the decomposition of the space of tensors having the same symmetries as the covariant derivative of the fundamental 2-form in [2] is given a classification scheme for the almostcontact metric manifolds containing 12 basic classes W_i, $i = 1, \ldots, 12$. The manifolds in the class $W_1 \oplus W_2 \oplus W_3 \oplus W_9$ are said to be coformal classes [4]. These manifolds are generated by Sasakian and cosymplectic manifolds by means of subgroups of the contact conformal group and they are the contact analogues of the conformally Keahler manifolds in the Hermitian geometry [3], [5].

Analogously to the Hermitian case [1] by making use of (2) we get a classification scheme for conformal manifolds with respect to the covariant derivative of the Lee form.

The 1-form θ on a manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ defined by

$$\theta = \frac{f^*(\xi)}{2n}.\eta + \frac{1}{2(n-1)}.f \circ \varphi$$

is called Lee form on M.

We call a manifold in the class W_i is in the subclass W_{ij}, $i = 1, 2, 3, 9$, $j = 1, \ldots, 9$ if θ is in the subspace Ω_j. The class corresponding to the subspace $\Omega_j \oplus \Omega_k$ will be denoted by $W_{ij} \oplus W_{ik}$.

We consider the class W_1. The defining condition for this class is $F = \eta \otimes (\eta \wedge \omega)$. The Lee form θ on a manifold in W_1 is given by $\theta = \omega \circ \varphi$. In [5] is proved that θ is contact closed, i.e. $hd\theta = 0$. Then Proposition 1 implies

(3) $$L_4(\theta) = L_5(\theta) = L_6(\theta) = 0.$$

The subclass W_1^0 of W_1 consists of all manifolds with θ closed, i.e. $d\theta = 0$. From Proposition 1 we have

(4) $$L_4(\theta) = L_5(\theta) = L_6(\theta) = L_8(\theta) = 0.$$

Theorem 1 and equations (3), (4) imply

THEOREM 2. The class W_1 and its subclass have the following subclasses:

$$W_1 = W_{11} \oplus W_{12} \oplus W_{13} \oplus W_{17} \oplus W_{18} \oplus W_{19},$$

$$W_1^0 = W_{11} \oplus W_{12} \oplus W_{13} \oplus W_{17} \oplus W_{19}.$$

5. THE SUBCLASSES W_{1j} OF THE CLASS W_1^0 AND THEIR EXACT CHARACTERIZATIONS. From theorems 1,2 we obtain the defining conditions for the subclasses
$W_{1j} = \{M(\varphi, \xi, \eta, g) \in W_1^0 : L(\theta) = L_j(\theta)\}, j = 1, \ldots, 9.$

A contact conformal transformation of the structure (φ, ξ, η, g) on an almost contact metric manifold M is defined (see [3]) by

(5) $\quad c(u, v) : M^{2n+1}(\varphi, \xi, \eta, g) \longrightarrow M^{2n+1}(\overline{\varphi} = \varphi, \ \overline{\xi} = \mathrm{e}^{-v}\xi, \overline{\eta} = \mathrm{e}^{v}\eta, \ \overline{g} = \mathrm{e}^{2u}hg + \mathrm{e}^{2v}\eta \otimes \eta),$

where u, v are differentiable functions on M. The set of all these transformations forms the contact conformal group G.

Let $M(\varphi, \xi, \eta, g)$ and $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be conformally related as in (5). The Levi-Civita connections ∇ and $\overline{\nabla}$ of the both structures are related (see [4]) by

$$
\begin{aligned}
2\overline{g}(\overline{\nabla}_x y, z) &= 2\mathrm{e}^{2u}g(\nabla_x y, z) + 2\mathrm{e}^{2u}\{du(x)g(y, z) + du(y)g(x, z) - du(z)g(x, y)\} + \\[4pt]
&\quad + 2\{[\mathrm{e}^{2v}dv(x) - \mathrm{e}^{2u}du(x)]\eta(y)\eta(z) + [\mathrm{e}^{2v}dv(y) - \mathrm{e}^{2u}du(y)]\eta(x)\eta(z) - \\[4pt]
&\quad - [\mathrm{e}^{2v}dv(z) - \mathrm{e}^{2u}du(z)]\eta(x)\eta(y)\} + (\mathrm{e}^{2v} - \mathrm{e}^{2u}).[2\eta(\nabla_x y)\eta(z) - \\[4pt]
&\quad - \eta(z).F(x, \xi, \varphi y) - \eta(y).F(x, \xi, \varphi z) - \eta(x).F(y, \xi, \varphi z) - \\[4pt]
&\quad - \eta(z).F(y, \xi, \varphi x) + \eta(y).F(z, \xi, \varphi x) + \eta(x).F(z, \xi, \varphi y)].
\end{aligned}
$$

(6)

We have the following

THEOREM ([3],[5]). The maximal subgroup of $G - G_1$ preserving the class W_1 consists of the transformations of type (5) satisfying the condition $du = 0$. The maximal subgroup G_1^0 of G preserving the subclass W_1^0 of W_1 consists of the transformations of G_1 satisfying the condition $dv(\xi) = 0$.

It follows from (6)

LEMMA. Let $M(\varphi, \xi, \eta, g)$ and $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be manifolds in the class W_1 and the structures $(\varphi, \xi, \eta, g), (\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ on M are conformally related as in (5) with a transformation $c(u, v) \in G_1$. Then the corresponding to both structures $\{\nabla, L(\theta), L_i(\theta)\}$ and $\{\overline{\nabla}, \overline{L}(\overline{\theta}), \overline{L}_i(\overline{\theta})\}$, $i = 1, 2, 3, 7, 9$ are related by

$$\overline{\nabla}_x y = \nabla_x y - \mathrm{e}^{2v-2u}\eta(x)\eta(y)h(\operatorname{grad} v) + [dv(x)\eta(y) + dv(y)\eta(x) - dv(\xi)\eta(x)\eta(y)]\xi \ ;$$

(7) $\overline{L}(\overline{\theta}) = L(\theta) - L(dv \circ \varphi) + \mathrm{e}^{2v-2u}.\theta(\mathrm{grad}\,v).\eta \otimes \eta;$

(8) $\overline{L}_i(\overline{\theta}) = L_i(\theta) - L_i(dv \circ \varphi) + \mathrm{e}^{2v-2u}.\theta(\mathrm{grad}\,v).\eta \otimes \eta, \quad i = 1,2,3,7,9$

for arbitrary x, y in T_pM, $p \in M$.

It is not difficult to verify that the sets

$$G_{1i}^0 = \{c(u,v) \in G : du = 0, dv(\xi) = 0, L(dv \circ \varphi) = L_i(dv \circ \varphi)\}, \quad i = 1,2,3,9$$

are subgroups of the contact conformal group.

THEOREM 3. The maximal subgroups of the contact conformal group preserving the subclasses W_{1i} of W_1^0 are G_{1i}^0, $\quad (i = 1,2,3,9)$.

PROOF. Let $M(\varphi, \xi, \eta, g)$ be in W_{1i}, $c(u,v) \in G_{1i}^0$ and $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) = c(u,v)(\varphi, \xi, \eta, g)$. It is known that $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is in W_i^0. It follows from (7), (8) and the defining conditions of G_{1i}^0 that $\overline{L}(\overline{\theta}) = \overline{L}_i(\overline{\theta})$, i.e. $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is in W_{1i}.

For the inverse let $M(\varphi, \xi, \eta, g)$ and $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be two manifolds in W_{1i} which are conformally related by a trasformation $c(u,v) \in G_1^0$. Then (7) and (8) imply $L(dv \circ \varphi) = L_i(dv \circ \varphi)$, i.e. $c(u,v) \in G_{1i}^0$, $i = 1,2,3,7,9$.

In [5] is prooved that an almost contact metric manifold M is in W_1^0 iff M is contact conformally related to a cosymplectic manifold by a transformation of the subgroup G_1^0. We get an analogously characterization for the classes G_{1i}^0 and the class of cosymplectic manifolds.

THEOREM 4. An almost contact metric manifold is in the class W_{1i} iff the structure of the manifold is locally conformal to a cosymplectic structure by a transformation of the group G_{1i}^0, $i = 1,2,3,7,9$.

PROOF. If $M(\varphi, \xi, \eta, g)$ is a cosymplectic in [3] is proved that by an arbitrary transformation of G_1^0 we obtain $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \in W_1^0$. Then (7) and (8) imply $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \in W_{1i}$ if $c(u,v) \in G_{1i}^0$.

For the inverse let $M(\varphi, \xi, \eta, g)$ be in W_{1i}. Since the Lee form $\theta = \omega \circ \varphi$ is closed, the Poincare's lemma implies that there exists locally function v such that $\theta = dv$. In [3] is proved that by a transformation $c(u,v)$ with the above function v and function $u : du = 0$ we obtain $M(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a cosymplectic manifold, i.e. $\overline{F} = 0$ and hence $\overline{\theta} = 0$. Thus (7) and (8) imply $L(dv \circ \varphi) = L_i(dv \circ \varphi)$ and hence the transformation $c(u,v)$ is in G_{1i}^0.

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